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LETTER TO THE EDITOR

Exact solubility of the D_n series

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Abstract. We show that the D_n series of a newly constructed set of models is completely integrable. In particular, the model described by D_4 is a non-critical extension of the three-state self-dual Potts model.

In an earlier paper (Pasquier 1986) we showed that any generalised RSOS model could be exactly solved at a certain critical point. The aim of this letter is to show that, for a certain class of them labelled by Dynkin diagrams, integrability can be extended out of the critical region.

By a generalised RSOS model, we mean an IRF model defined in terms of a set of m even heights and a set of n odd heights lying on two interpenetrating sublattices. A matrix A_{ij}, i labelling even heights and j odd heights, is defined with matrix elements being equal to 1 or 0 according to whether heights i and j can or cannot be adjacent. We defined the Cartan matrix of the problem to be the $(n+m)\times(n+m)$ matrix

$$C=\begin{pmatrix} 2 & -A \\ -A^{\mathrm{T}} & 2 \end{pmatrix}$$

and built a Temperley and Lieb algebra (Temperley and Lieb 1971) from the vector with smallest eigenvalue of C.

A particularly interesting class of models are those for which the largest eigenvalue of AA^{T} is smaller than 4. Then C is positive and corresponds to the Cartan matrix of a simply laced Lie algebra (A_n, D_n, E_6, E_7, E_8). The set of models corresponding to A_n are those considered by Andrews *et al* (1984) with their parameter r equal to $n+1$.

In order to define the notation, let us review the case of A_n. The heights l_1 can take any integer value $1\leqslant l_1\leqslant n$ and two adjacent heights must differ by unity: $|l_1-l_j|=1$. Following Andrews *et al* (1984) we adopt the following notation for weights:

$$\begin{aligned} &W(l,l+1|l-1,l)=W(l,l-1|l+1,l)=\alpha_l\\ &W(l+1,l|l,l-1)=W(l-1,l|l,l+1)=\beta_l\\ &W(l+1,l|l,l+1)=\gamma_l\\ &W(l-1,l|l,l-1)=\delta_l. \end{aligned}\tag{1}$$

One can then write the equations resulting from the star-triangle relations (STR):

$$\sum_g W(a,c|a,g)\,W'(a,g|f,e)\,W''(g,c|e,d)$$
$$=\sum_g W''(a,b|f,g)\,W'(b,c|g,d)\,W(g,d|f,e).\tag{2}$$

One ends up with the following set of equations (Andrews *et al* 1984, formula (1.4.7))

$$\beta_l\alpha'_l\beta''_l+\gamma_l\delta'_{l+1}\gamma''_l=\delta_{l+1}\gamma'_l\delta''_{l+1}+\beta_{l+1}\alpha'_{l+1}\beta''_{l+1} \tag{3a}$$

$$\beta_l\gamma'_{l-1}\beta''_l+\gamma_l\alpha'_l\gamma''_l=\alpha_l\gamma'_l\alpha''_l \tag{3b}$$

$$\beta_l\alpha'_l\delta''_l+\gamma_l\delta'_{l+1}\beta''_l=\alpha_l\beta'_l\delta''_{l+1} \tag{3c}$$

$$\delta_l\alpha'_l\delta''_l+\beta_l\delta'_{l+1}\beta''_l=\alpha_l\delta'_l\alpha''_l \tag{3d}$$

$$\delta_l\gamma'_{l-1}\beta''_l+\beta_l\alpha'_l\gamma''_l=\alpha_l\beta'_l\gamma''_{l-1} \tag{3e}$$

$$\alpha_{l+1}\alpha'_l\alpha''_{l+1}=\alpha_l\alpha'_{l+1}\alpha''_l \tag{3f}$$

$$\alpha_{l+1}\beta'_{l+1}\beta''_l=\alpha_l\beta'_l\beta''_{l+1} \tag{3g}$$

and three other equations obtained by interchanging unprimed and double prime symbols. l takes values $1,\ldots,n-1$ in the first equation, the values $2,\ldots,n-2$ in the next four equations and $2,\ldots,n-3$ in the last two. In the first equations, the terms $\beta_1\alpha'_1\beta''_1$, $\beta_n\alpha'_n\beta''_n$ should be deleted. For the unrestricted model where l can take all integer values, a general solution of (3) is given by

$$\begin{aligned}
\alpha_l&=\theta_1(\mu-u)/\theta_1(\mu)\\
\beta_l&=\theta_1(u)(\theta_1(\mu))^{-1}\left(\frac{\theta_1((l-1)\mu+W_0)\theta_1((l+1)\mu+W_0)}{\theta_1(l\mu+W_0)}\right)^{1/2}\\
\gamma_l&=\theta_1(l\mu+u+W_0)/\theta_1(l\mu+W_0)\\
\delta_l&=\theta_1(l\mu-u+W_0)/\theta_1(l\mu+W_0)
\end{aligned} \tag{4}$$

where θ_1 is the odd elliptic theta function and μ and W_0 are any real numbers.

Equation (2) is fulfilled iff $u''=u'+u$ in the expressions of W, W', W''.

For the restricted model where l is restricted to the internal $1,\ldots,n$ in order to fulfil (3a) for $l=1$ and $l=n-1$ one is led to set $W_0=0$, $(n+1)\mu=\pi$ so that $\beta_1=\beta_n=0$.

Information about the model is schematically represented in figure 1. Two heights can be adjacent only if they are linked on the diagram. For example, heights 0 and $\bar{0}$ lying on the even sublattice can only be adjacent to height 1 on the odd sublattice. There are two kinds of weights. Those for which l does not take the value 0 or $\bar{0}$ keep the notations of the RSOS model. Weights containing 0 or $\bar{0}$:

$$\begin{aligned}
&W(1,2|0,1)=W(1,2|\bar{0},1)=W(1,0|2,1)=W(1,\bar{0}|2,1)=\alpha\\
&W(1,\bar{0}|0,1)=W(1,0|\bar{0},1)=\bar{\alpha}\\
&W(2,1|1,0)=W(0,1|1,2)=W(2,1|1,\bar{0})=W(\bar{0},1|1,2)=\beta\\
&W(0,1|1,\bar{0})=W(\bar{0},1|1,0)=\bar{\beta}\\
&W(1,0|0,1)=W(1,\bar{0}|\bar{0},1)=\gamma\\
&W(0,1|1,0)=W(\bar{0},1|1,\bar{0})=\delta.
\end{aligned} \tag{5}$$

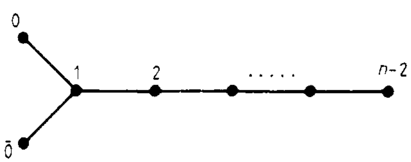

Figure 1. D_n.

We have imposed diagonal reflection symmetry conditions:

$$W(a, b|c, d) = W(d, b|c, a) = W(a, c|b, d) \tag{6}$$

and respected the $\mathbb{Z}_2$ symmetry of figure 1.

There are two kinds of STR, those involving weights of the first kind only, which will be of the same kind of those occurring in equation (3) and others involving weights of the second kind:

$$\alpha_2\alpha'\alpha_2'' = \alpha\alpha_2'\alpha'' \tag{7a}$$

$$\beta\beta_2'\alpha_2'' = \beta_2\beta'\alpha'' \tag{7b}$$

$$\gamma_1\delta_2'\gamma_1'' + 2\beta\alpha'\beta'' = \delta_2\gamma_1'\delta_2'' + \beta_2\alpha_2'\beta_2'' \tag{7c}$$

$$\gamma_1\alpha'\gamma_1'' + \beta(\gamma' + \bar{\alpha}')\beta'' = \alpha\gamma_1'\alpha'' \tag{7d}$$

$$\gamma_1\delta_2'\beta'' + \beta\alpha'(\delta'' + \bar{\beta}'') = \alpha\beta'\delta_2'' \tag{7e}$$

$$\gamma_1\alpha'\beta'' + \beta\bar{\alpha}'\bar{\beta}'' + \beta\gamma'\delta'' = \gamma\beta'\alpha'' \tag{7f}$$

$$\gamma_1\alpha'\beta'' + \beta\bar{\alpha}'\delta'' + \beta\gamma'\bar{\beta}'' = \bar{\alpha}\beta'\alpha'' \tag{7g}$$

$$\beta\delta_2'\beta'' + \delta\alpha'\delta'' + \bar{\beta}\alpha'\bar{\beta}'' = \alpha\delta'\alpha'' \tag{7h}$$

$$\beta\delta_2'\beta'' + \delta\alpha'\bar{\beta}'' + \bar{\beta}\alpha'\delta'' = \alpha\bar{\beta}'\alpha'' \tag{7i}$$

$$\beta\alpha'\beta'' + \delta\gamma'\delta'' + \bar{\beta}\bar{\alpha}'\bar{\beta}'' = \gamma\delta'\gamma'' \tag{7j}$$

$$\beta\alpha'\beta'' + \delta\bar{\alpha}'\delta'' + \bar{\beta}\gamma'\bar{\beta}'' = \bar{\alpha}\delta'\bar{\alpha}'' \tag{7k}$$

$$\beta\alpha'\beta'' + \delta\gamma'\bar{\beta}'' + \bar{\beta}\bar{\alpha}'\delta'' = \bar{\alpha}\bar{\beta}'\gamma''. \tag{7l}$$

STR of the first kind will automatically be fulfilled if we adopt parametrisation (4) for the indexed parameters. The condition $\beta_{n-2} = 0$ then implies

$$(n-1)\mu + W_0 = \pi. \tag{8}$$

From the rational case (Pasquier 1986), we are led to take

$$W_0 = \pi/2 \qquad \mu = \pi/2(n-1). \tag{9}$$

It remains to find the unindexed weights in (7). The calculation is closely related to the one performed by Akutsu *et al* (1986). From their work we deduce the following. Comparing (7*a*) and (3*f*)

$$\alpha = \alpha_1 \tag{10}$$

(7*b*, *c*) and (3*g*, *a*)

$$\beta = \frac{1}{\sqrt{2}}\beta_1 \tag{11}$$

(7*d*) and (3*b*)

$$\tfrac{1}{2}(\gamma + \bar{\alpha}) = \gamma_0 \tag{12}$$

(7*e*) and (3*c*)

$$\delta + \bar{\beta} = \delta_1. \tag{13}$$

Adding and subtracting (7*f*, *g*) we recover (3*e*) and

$$\beta(\gamma - \bar{\alpha})(\delta - \bar{\beta}) = (\gamma - \bar{\alpha})\beta\alpha. \tag{14}$$

Similarly, adding and subtracting ($7h, i$) we recover ($3d$) and

$$(\delta-\bar{\beta})\alpha(\delta-\bar{\beta})=\alpha(\delta-\bar{\beta})\alpha. \tag{15}$$

Comparing (14) and (15) with ($3g$) and ($3f$), respectively, we deduce

$$\begin{aligned}&\delta-\bar{\beta}=\alpha_1\\&\tfrac{1}{2}(\gamma-\bar{\alpha})=\Gamma\beta_1.\end{aligned} \tag{16}$$

Substituting (11), (12), (13) and (16) in ($7j, k, l$) we see that the term linear in Γ has to vanish. This leads to

$$\alpha_1\beta_1\delta_1=\beta_1\alpha_1\gamma_0+\gamma_0\delta_1\beta_1 \tag{17}$$

and two new equations determine Γ:

$$\begin{aligned}&\gamma_0\alpha_1\gamma_0+\Gamma^2\beta_1\delta_1\beta_1=\alpha_1\gamma_0\alpha_1\\&\delta_1\gamma_0\delta_1+(1-\Gamma^2)\beta_1\alpha_1\beta_1=\gamma_0\delta_1\gamma_0.\end{aligned} \tag{18}$$

Using equation (1.13) of Gaudin (1983) (17) is easily deduced and the value of Γ is

$$\Gamma=\theta_2^2(\mu)\left(\frac{1}{\theta_2(2\mu)\theta_2^3(0)}\right)^{1/2}. \tag{19}$$

So we have obtained the expressions of unindexed weights that solve (7):

$$\begin{aligned}&\beta=\frac{1}{\sqrt{2}}\beta_1\\&\bar{\beta}=\tfrac{1}{2}(\delta_1-\alpha_1)\\&\delta=\tfrac{1}{2}(\delta_1+\alpha_1)\\&\gamma=(\gamma_0+\Gamma\beta_1)\\&\bar{\alpha}=(\gamma_0-\Gamma\beta_1).\end{aligned} \tag{20}$$

Indexed weights on the RHS are given by (4).

In conclusion, we have shown that the D_n models of a newly discovered series are completely integrable. D_4 describes a non-critical extension of the three-state self-dual Potts model to which it reduces in the trigonometric limit. Clearly the E series can be handled similarly. We hope to report on them and give the local density expressions in a further publication. These models should describe universality classes of unitary conformal theories (Pasquier 1986). Let us end with a question: can we extend the construction to models with a general Cartan matrix C (true for the extended Dynkin diagrams)? Is there a classification for them?

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